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Stochastic formulation of the renormalization group: supersymmetric structure and topology of the space of couplings

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Abstract
The exact or Wilson renormalization group equations can be formulated as a functional Fokker–Planck equation in the infinite-dimensional configuration space of a field theory, suggesting a stochastic process in the space of couplings. Indeed, the ordinary renormalization group differential equations can be supplemented with noise, making them stochastic Langevin equations. Furthermore, if the renormalization group is a gradient flow, the space of couplings can be endowed with a supersymmetric structure a la Parisi–Sourlas. The formulation of the renormalization group as supersymmetric quantum mechanics is useful for analysing the topology of the space of couplings by means of Morse theory. We present simple examples with one or two couplings.

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1. Introduction

The concept of the renormalization group arose in quantum electrodynamics and was soon applied to other quantum field theories and later to critical phemomena. With the application of the renormalization group (RG) to several couplings, it became clear that it could have interesting features as a system of autonomous ordinary differential equations and, in particular, that the topology of the RG trajectories should play a crucial role [1]. The simplest topologies correspond to trajectories that follow the gradient of some potential. This gradient RG flow hypothesis was discussed in [2]. With the generalization of this hypothesis to the existence of an irreversible RG function, after Zamolodchikov c-theorem in two dimensions [3], it has been the subject of numerous papers (as a representative sample, see [4–6]).

However, the study of the topology of the space of couplings of a field theory is still in its infancy. Even under the assumption of gradient RG flow (or irreversible RG function) very few general results exist. In two dimensions the problem has received more attention, because of the powerful methods provided by conformal symmetry and the connection with string theory.

A particularly interesting development in this regard is the relation with supersymmetric quantum mechanics (SUSY QM) and Morse theory, two concepts which were connected in Witten's seminal papers [7], independently of the RG.

Das, Mandal and Wadia proposed the connection of two-dimensional RG equations with stochastic quantization and supersymmetric quantum mechanics in the context of string theory [8]. The motivation was that two-dimensional quantum field theories are the basis of first quantized ('classical') string theory and the field equations are given by conformal invariance, that is, by the vanishing of the β-functions corresponding to low-energy fields, which play the role of couplings. Therefore, the interpolation between RG fixed points, given by the RG flow, represents the transition between string theory solutions, and a potential for the flow is also a low-energy string potential. In this context, it is natural to introduce supersymmetry in the space of couplings, which are now low-energy fields. The underlying supersymmetry of stochastic quantization had been discovered earlier by Parisi and Sourlas [9] (for a systematic treatment, see [10]). In the string theory context, it is natural to assume that the fields have a stochastic character and, in fact, this character corresponds to quantized string field theory, that is, to *second-quantized* string theory.

A different point of view was adopted by Vafa [11], regarding the topology of the space of two-dimensional quantum field theories as given by Zamolodchikov's c-function when considered as a Morse function.

We adopt here a more general standpoint: the field theories need not be two-dimensional and, hence, need not have any relation with string theory. Supersymmetry in the space of couplings is just a convenient mathematical structure to study the topological structure of this space, following the spirit of Witten's paper 'Supersymmetry and Morse theory' [7]. Nevertheless, one can also provide a rationale for an interpretation of the RG in connection with stochastic quantization, independent of string theory. It arises from the *exact* formulation of the RG (including every coupling) which gives rise to a functional Fokker–Planck equation.

So we begin by describing the exact RG and describing its functional equation as a Fokker–Planck equation. Then we restrict ourselves to the usual RG in a finite space of couplings and examine when it can be considered a gradient flow. In this regard, one must take into account the freedom in the choice of metric as well as the freedom in the choice of coordinates. Next, assuming a gradient RG flow, we make the connection with SUSY QM. Finally, we review Witten's reinterpretation of Morse theory as SUSY QM and show some applications of Morse theory to simple examples of RG flow.

2. The exact renormalization group

When one says that one is interested in defining the theory at the scale L, one is, first of all, redefining the field ϕ to that scale, by means of an averaging with a suitable kernel:

$$\phi_L(\boldsymbol{r}) = \int K_L(\boldsymbol{r}-\boldsymbol{x})\phi(\boldsymbol{x}). \tag{1}$$

This is called 'coarse graining'. Customary kernels are the Gaussian kernel $K_L(\boldsymbol{x}-\boldsymbol{y}) = \exp(-\pi|\boldsymbol{x}-\boldsymbol{y}|^2/L^2)$ or the 'top-hat' kernel $K_L(\boldsymbol{x}-\boldsymbol{y}) = 1-\theta(|\boldsymbol{x}-\boldsymbol{y}|^2/L^2-1)$ (where θ is the step function). The first one belongs to the type of 'smooth kernels', that is, which are regular functions, whereas the second one does not (for further explanation, see [12]). We just demand that the kernel has an inverse. In Fourier space, the coarse graining convolution adopts a simple multiplicative form

$$\phi_L(\boldsymbol{k}) = K_L(\boldsymbol{k})\phi(\boldsymbol{k}). \tag{2}$$

Hence, an inverse exists if $K_L(\boldsymbol{k})$ has no zeros.

Let us now examine the simple case of the evolution of a Gaussian probability distribution under a change of L. The most general Gaussian probability distribution can be written as

$$\mathcal{P}[\phi] = \exp\left\{-\frac{1}{2}\phi \cdot G^{-1} \cdot \phi\right\} = \exp\left\{-\frac{1}{2}\frac{\phi_L}{K_L} \cdot G^{-1} \cdot \frac{\phi_L}{K_L}\right\} \tag{3}$$

where $G(|\boldsymbol{x}-\boldsymbol{y}|)$ is the covariance function (the free propagator in QFT) and we use condensed notation, valid in 'real' or Fourier space. This evolution can be considered trivial: the coarse-grained field has a variance depressed in the high wavenumbers $G_L(\boldsymbol{k}) = K_L(\boldsymbol{k})^2 G(\boldsymbol{k})$.

The evolution of the non-Gaussian part of the probability distribution with L is more interesting and, not surprisingly, the calculation leading to it is rather involved: it is the general form of the Wilson or exact RG. The exact formulation of the RG was proposed by Wilson [1] and it has been afterwards the subject of profound studies. We refer the interested reader to the literature [12, 13]. We are mainly interested here in the fact that the equation for the evolution of the non-Gaussian part of the probability distribution can be written as a linear functional partial differential equation [1, 13, 14]:

$$\frac{\partial}{\partial L}\,\mathrm{e}^{-V_L} = -\frac{1}{2}\frac{\partial G_L}{\partial L}\frac{\delta^2}{\delta\phi_L^2}\,\mathrm{e}^{-V_L} \tag{4}$$

where V_L is the scale-dependent effective potential. This equation is the simplest form of a functional Fokker–Planck equation, namely, a functional heat or diffusion equation (a general functional Fokker–Planck equation including a term with a first functional derivative results if the Gaussian part is included [1]).

The essence of coarse graining as we have introduced it is that it removes the small-scale information in a sort of diffusion process governed by the usual equations of stochastic dynamics. In particular, the non-Gaussian part of the probability distribution tends to vanish in the process, whereas the Gaussian part tends to a fixed form with only low-$\boldsymbol{k}$ wavenumbers (in the limit $L \to \infty$, only the constant field $\boldsymbol{k} = 0$).

The mentioned RG-induced stochastic process takes place in the space of field configurations and therefore the Fokker–Planck equation is satisfied by the probability distribution as a function of the field configuration. We can consider this probability distribution parametrized by an infinite set of coupling constants in the usual way. The RG-induced evolution in the space of coupling constants is deterministic, in principle. However, if we take into account that operational forms of the RG can only consider a finite number of couplings and, therefore, need to truncate the whole space in some way, we may appreciate that the consequent loss of information must somehow be added to the inherent loss of information pertaining to small scales. In fact, both types of information are intertwined, since the removal of small-scale degrees of freedom leads to the removal of their couplings. We conclude that a stochastic process in the space of couplings follows from the very nature of the implementation of the RG. We will take advantage of this picture in the following.

3. The RG as a gradient flow

Here we leave the exact RG and we consider the classical formulation of the RG as a system of autonomous first order ordinary differential equations (ODE) for a *finite* set of couplings g_i (possibly, only one):

$$\frac{\mathrm{d}g_i}{\mathrm{d}\tau} = \beta_i(g) \tag{5}$$

where we use a nondimensional RG parameter τ, equivalent to the logarithm of the normalized relevant scale (e.g. to the logarithm of the coarse-graining scale $\tau = \log(L/L_0)$). Furthermore, we consider the situation close to a fixed point g_i^*:

$$\beta_i(g) = \beta_i(g^*) + \left.\frac{\partial \beta_i}{\partial g_j}\right|_{g^*} (g_j - g_j^*) + \cdots \tag{6}$$

with $\beta_i(g^*) = 0$, so the behaviour of the RG is given by the linear terms, namely, by the matrix $\Delta_{ij} := \frac{\partial\beta_i}{\partial g_j}\big|_{g^*}$. This matrix is called the dimension matrix because, when it is diagonalizable $(\Delta_{ij} \to \Delta_i \delta_{ij})$, the eigenvalues Δ_i give the simple solution $g_i = g_i(0)e^{\Delta_i \tau} = g_i(L_0)(L/L_0)^{\Delta_i}$ (taking $g^* = 0$, for simplicity), so they are proper dimensions. We expect the dimensions to be *real* positive or negative numbers, not necessarily integers. So the generic fixed point is *hyperbolic* (a saddle point).

We may look for general conditions implying that a fixed point has real dimensions. Obviously, if the dimension matrix is symmetric it can be diagonalized with real eigenvalues. This is a necessary and sufficient condition, but without further meaning. A sufficient condition is that the matrix of derivatives of the beta function is symmetric in a whole neighbourhood of g^*, namely, $\frac{\partial\beta_i}{\partial g_j} = \frac{\partial\beta_j}{\partial g_i}$. It means that the curl of β_i vanishes, so that it is the gradient of some function $V(g)$: $\beta_i = \frac{\partial V}{\partial g_i}$; this is called a gradient flow [2]. It follows that $\frac{\mathrm{d}V}{\mathrm{d}\tau} = \beta^2 \geqslant 0$, that is, V is a monotonic (Lyapunov) function of the system of ODE.

Geometrically speaking, the gradient flow is orthogonal to the (hyper)surfaces of constant V. This orthogonality depends on a metric, which has been taken to be Euclidean by default. In fact, covariance demands that the gradient flow condition be written as $\beta^i = G^{ij}\partial_j V$, where the metric can be arbitrary. So if we are given a flow the question of whether it is a gradient flow or not is somewhat ambiguous and can be interpreted as the question of whether there can be found a metric that makes it a gradient flow. Now, it is easy to convince oneself that if we allow for an arbitrary metric we can always conclude that a flow is gradient near a fixed point if and only if the dimensions are real. Of course, we should then ask for a *natural* metric and that it be *globally* defined.

Therefore, we can express the gradient flow condition in an intrinsic form: since the RG flow is given by a vector field β on a manifold, it is a gradient flow if for some metric G in the manifold the 1-form $\theta = G(\beta)$ is exact, namely, $\theta = \mathrm{d}V$. Furthermore, we expect to have a natural metric. Indeed, there is a natural metric in the space of coupling constants when these are considered as statistical parameters: the Fisher metric of estimation theory [15]. The quest for RG gradient flows with this metric has already had partial success [16, 17]. Furthermore, the relation of Fisher metric with entropy (or information) constitutes the basis for an extension of the time-irreversibility H-theorems to irreversibility under scale transformations (that is, under the RG) [5]. We must also mention that in two dimensions there is another candidate for a natural metric, namely, Zamolodchikov's metric [3]. Intriguingly, the Fisher metric (valid in any dimension) and Zamolodchikov's metric adopt somewhat similar expressions [6].

3.1. Freedom in the choice of coordinates: scaling fields

We have mentioned that the RG must act covariantly in the space of couplings; in other words, we are free to choose coordinates in this space, redefining the couplings. This large freedom implies in particular that we can always make the RG a gradient flow by linearizing the β functions: the corresponding coordinates are called scaling fields (scaling is homogeneous in these coordinates) [2]. The possibility of linearizing a flow is in fact a general result of the theory of ODE, in which it is called Poincaré theorem [18]. In the quantum field theory the scaling fields are to be identified with the bare couplings. The bare couplings indeed scale with their naive dimensions whereas the behaviour of renormalized couplings under a change of scale is given by the beta functions, including ‘anomalous’ dimensions.

The simplest example is perhaps the RG for the theory $\lambda\phi^4$ (in dimension $D < 4$). The one-loop RG equation can be written as [10]

$$\frac{d\lambda}{d\tau} = -\mu\frac{d\lambda}{d\mu} = \lambda - \lambda^2 \tag{7}$$

after linearly redefining both τ and the coupling λ to make numerical coefficients equal to 1. These redefinitions place the fixed points at $\lambda = 0, 1$. The solution of this equation with the condition $\lambda(1) = \lambda_0 \in (0, 1)$ is

$$\lambda = \frac{\lambda_0}{\mu + \lambda_0(1-\mu)}. \tag{8}$$

It gives the flow between the two fixed points. In the UV limit $\mu \to \infty$, $\lambda \to 0$ but $\mu\lambda$ stays finite. We can define the scaling coupling

$$\lambda_b = \lim_{\mu\to\infty}(\mu\lambda) = \frac{\lambda_0}{1-\lambda_0}.$$

Indeed

$$\tilde{\lambda} = \frac{\lambda}{1-\lambda} \tag{9}$$

is the coordinate transformation that linearizes the RG:

$$\mu\frac{d\tilde{\lambda}}{d\mu} = -\tilde{\lambda} \quad\Rightarrow\quad \tilde{\lambda} = \frac{\lambda_b}{\mu}. \tag{10}$$

Note that in the scaling coordinate the IR fixed point is located at $\tilde{\lambda} \to \infty$.

Let us remark that transformation (9) is projective (an RP^1 mapping). This is no coincidence: in general, one-loop RG equations implement projective transformations of the couplings and the real projective space is the natural compactification of the space of couplings [5]. While in scaling coordinates the nontrivial fixed points (that is, other than the one at the origin) are located at infinity and there is a trivial quadratic potential for the RG flow, in coordinates that cover the nontrivial fixed points the potential becomes nontrivial. Furthermore, projective space is homogeneous with its natural metric. Hence, this metric is also the natural metric for covariant gradient RG flow.

4. Stochastic RG and SUSY in the space of couplings

Let us assume that the RG is a gradient flow in a finite-dimensional space of couplings, and that the state of the system is represented by a probability distribution on this space, as remarked in section 2. Hence, we can derive interesting consequences.

The implementation of the RG leads to loss of information on the couplings, so that the exact state given by the infinite set of coupling constants becomes a probability distribution $P(g^i)$ over a finite set of couplings (as remarked at the end of section 2) and the RG evolution of these couplings can be represented by adding stochastic components to the β-functions. This makes the RG equations Langevin equations:

$$\frac{dg^i}{d\tau} = \beta^i(g) + \eta^i \qquad \langle\eta^i(\tau)\eta^j(\tau')\rangle = G^{ij}(g)\delta(\tau-\tau'). \tag{11}$$

The noise is assumed to be white and we have introduced the metric for covariance. $P(g^i)$ satisfies a Fokker–Planck equation, associated with the preceding Langevin equations [10].

Assuming that $\beta_i(g) = \partial_i V(g)$ (corresponding to a purely dissipative Langevin equation), the Fokker–Planck equation can be expressed as a Schrödinger equation in imaginary time with a *Hermitian* Hamiltonian.

Let us first express the Fokker–Planck equation as

$$\frac{\partial \mathcal{P}(g,\tau)}{\partial \tau} = -H_{\rm FP}\mathcal{P}(g,\tau) = \frac{1}{2}\nabla^2\mathcal{P}(g,\tau) - \nabla_i(\beta^i\mathcal{P}(g,\tau)). \tag{12}$$

Then, the Hermitian Hamiltonian is

$$\tilde{H}_{\rm FP} = {\rm e}^{-V} H_{\rm FP}\,{\rm e}^{V} = -\tfrac{1}{2}(\nabla^2 - (\nabla V)^2 - \nabla^2 V) = \tfrac{1}{2}A_i^\dagger A_i \tag{13}$$

with $\boldsymbol{A} = \nabla - \nabla V$. This Hamiltonian operates on states $|g,\tau\rangle = {\rm e}^{-V}\mathcal{P}(g,\tau)$, the equilibrium state being $\langle g,\tau|0\rangle = {\rm e}^{V}$, such that $\boldsymbol{A}|0\rangle = 0$. 'Excited states' are produced by the action of $\boldsymbol{A}^\dagger$.

4.1. Connection with SUSY QM

Let us now summarize Das, Mandal and Wadia's procedure to represent the probability distribution $P(g,\tau)$ as SUSY QM, following Parisi–Sourlas's [9] and Witten's [7] methods:

1. Introduce fermionic coordinates $\psi^i(\tau)$ and $\bar{\psi}^i(\tau)$ (Grassmann variables), such that $(\psi^i)^2 = [\bar{\psi}^i]^2 = 0$, $\{\psi^i, \bar{\psi}^j\} = G^{ij}$.
2. Introduce supercharges $Q = \sum_i \psi^i(\nabla_i + \beta_i)$ and $\bar{Q}$. Note that $Q^2 = (\bar{Q})^2 = 0$ if and only if $\beta_i = \partial_i V$, that is, if the RG is a gradient flow.
3. Complete the SUSY algebra with the supersymmetry Hamiltonian

$$H = \tfrac{1}{2}(Q\bar{Q} + \bar{Q}Q) = \tfrac{1}{2}(-\nabla^2 + G^{ij}\partial_i V\partial_j V + \nabla_i\nabla_j V[\psi^i, \bar{\psi}^j]). \tag{14}$$

This is just the SUSY generalization of the Hamiltonian (13).

The Euclidean action corresponding to the preceding Hamiltonian is [7, 8]

$$\mathcal{S} = \frac{1}{2}\int {\rm d}\tau \left[G_{ij}\left(\frac{{\rm d}g^i}{{\rm d}\tau}\frac{{\rm d}g^j}{{\rm d}\tau} + \partial^i V\partial^j V\right) - G_{ij}\bar{\psi}^i\frac{{\rm d}g^k}{{\rm d}\tau}\nabla_k\psi^j + \frac{1}{4}R_{ijkl}\bar{\psi}^i\psi^k\bar{\psi}^j\psi^l + \nabla_i\nabla_j V\bar{\psi}^i\psi^j\right] \tag{15}$$

in which appear the Riemann curvature tensor, etc. This is the action of a *one-dimensional* $N = 2$ supersymmetric nonlinear σ-model. It can be written in terms of the supercoordinate $\phi(\tau) = g(\tau) + {\rm i}\theta\psi(\tau) - {\rm i}\bar{\psi}(\tau)\bar{\theta} + \bar{\theta}\theta\nabla V$:

$$\mathcal{S} = \int {\rm d}\tau\,{\rm d}\bar{\theta}\,{\rm d}\theta\left(\frac{1}{2}D\phi\cdot\overline{D\phi} - V(\phi)\right) \tag{16}$$

where $D = \partial_\theta - \bar{\theta}\partial_\tau$, and V is the superpotential.

The bosonic part of the action is

$$\mathcal{S} = \frac{1}{2}\int {\rm d}\tau\, G_{ij}\left(\frac{{\rm d}g^i}{{\rm d}\tau}\frac{{\rm d}g^j}{{\rm d}\tau} + \partial^i V\partial^j V\right). \tag{17}$$

The analysis of the minima of this action has important consequences for the topology of the space of couplings. Indeed, it is easy to see [7] that the minima occur for

$$\frac{{\rm d}g^i}{{\rm d}\tau} \pm \partial^i V = 0 \tag{18}$$

which defines the gradient flow (in either direction).

5. Morse theory and topology of the space of couplings

Morse theory [19] extracts topological information on a manifold from the knowledge of the critical points of some function on the manifold. Conversely, if the topology of the manifold is known, one can use it to deduce the existence and properties of the critical points of a function. Morse theory is generally applied to finite-dimensional manifolds but it has also been used in some infinite-dimensional spaces [11].

The question in our case is what space we should consider. The basic infinite-dimensional space seems to be the projective space of probability distributions, its projective character coming from the normalization of the probability distribution [17]. However, the RG flow that we are considering operates in a finite-dimensional subspace. As we mentioned in section 3, the natural geometry seems to correspond to a real projective space RP^n (see also [17]). The topology of the real projective space is well known, so we can deduce the properties of the critical points of any potential defined on that space. In order to see how to proceed, let us review Witten's reinterpretation of Morse theory as SUSY QM [7].

Witten considers the fermion coordinates ψ^i and $(\psi^*)^i$ as operators on the exterior algebra acting by interior and exterior multiplication, respectively. The basic objects in the algebraic topological theory by means of de Rham cohomology are the exterior derivative d, its adjoint d^* and the Hodge Laplacian $\Delta = (d + d^*)^2$. The supersymmetry operators Q and Q^* are then interpreted as new exterior derivatives obtained from d and d^* by conjugation with the exponential of a function V, namely, $Q = \mathrm{e}^{-V} d\, \mathrm{e}^{V}$ and $Q^* = \mathrm{e}^{V} d^*\, \mathrm{e}^{-V}$. Thus the Hamiltonian (14) is the transform of the Hodge Laplacian. It is easy to prove that this transformation is an isomorphism of the exterior algebra, so the algebraic topological properties are left unchanged by it. Furthermore, if we consider the classical limit, that is, when the noise fluctuations are negligible and the classical equations (18) hold, the isomorphism is still valid, so we deduce that the topological properties can be extracted from the critical points of V.

Morse theory assumes that the critical points of V are non-degenerate, that is, the Hessian determinant is nonvanishing at them. The topological information is encoded in the *index* of V at the critical points, which is defined as the number of negative eigenvalues of the Hessian matrix. In fact, the Morse lemma shows that in a neighbourhood of a critical point exist local coordinates such that the function is a quadratic form (of course, related to the scaling coordinates of section 3) and, furthermore, that the coefficients can be made to be ± 1. Therefore, the only topological information is in the relative number of negative and positive coefficients, that is, the index. One then associates with V and its critical points the Morse polynomial

$$M(V) = \sum_{P_i} t^{n_i} \tag{19}$$

where P_i are the critical points and n_i are their respective indices. The topology of the manifold enters via the Poincaré polynomial

$$P = \sum_{i=0}^{n} b_i t^i \tag{20}$$

where $b_i = \dim H^i$ are the Betti numbers, defined as the dimensions of the cohomology groups. The fundamental result of Morse theory (Morse inequalities) is that $M(V) \geqslant P$ and, moreover,

$$M(V) - P = (1 + t)Q \tag{21}$$

where Q is a polynomial with positive coefficients. A function V for which $M(V) = P$ is called a perfect Morse function. For every compact finite-dimensional manifold one can find a perfect Morse function.

An interesting consequence of equation (21) occurs for $t = -1$, namely,

$$M(V)(-1) = P(-1) = \sum_{i=0}^{n} (-1)^i b_i = \chi \tag{22}$$

that is, the Euler–Poincaré characteristic. Therefore, the Poincaré–Hopf index theorem on the zeros of a vector field [20] of gradient type is a particular case of equation (21) (note that V need not be a perfect Morse function).

6. RG gradient flows with one or two couplings

An elementary application is the theory $\lambda\phi^4$ considered in section 3. The potential V in the scaling coordinate seems to be just $V = \tilde{\lambda}^2/2$ but we must account for the metric of $RP^1 = S^1/Z_2 \simeq S^1$ (in general, $RP^n = S^n/Z_2$, where the Z_2 factor is to identify antipodal points). The metric in this coordinate is $\mathrm{d}s^2 = \mathrm{d}\tilde{\lambda}^2/(1+\tilde{\lambda}^2)^2$. So the correct potential is

$$V = \frac{\tilde{\lambda}^2}{2(1+\tilde{\lambda}^2)} \tag{23}$$

which coincides with $\tilde{\lambda}^2/2$ when $\tilde{\lambda} \ll 1$ and has a finite limit when $\tilde{\lambda} \to \infty$. Note that the critical points of V are $\tilde{\lambda} = 0, \infty$, that is, both RG fixed points. The Morse polynomial is simply $M(V) = 1 + t$. Naturally, the Poincaré polynomial of RP^1 is also $P = 1 + t$ so V is a perfect Morse function.

The function $\beta(\lambda)$ of the $\lambda\phi^4$ theory at more than one loop order is a higher degree polynomial, so it may have more than two fixed points and then corresponds to a potential V with several extrema. If this happens, $M(V)$ also becomes a higher degree polynomial, so V is no more a perfect Morse function. At any rate, the validity of perturbation theory for finding the additional nontrivial fixed points, the only important being the first one (at any loop order), is questionable.

A somewhat less elementary application is the theory for tricritical behaviour $r\phi^2 + \lambda\phi^4 + g\phi^6$ (in dimension $3 \leqslant D < 4$) [5]. The RG equations for the relevant bare couplings (the scaling coordinates) are just

$$\frac{\mathrm{d}\tilde{r}}{\mathrm{d}\tau} = \varphi\tilde{r} \tag{24}$$

$$\frac{\mathrm{d}\tilde{\lambda}}{\mathrm{d}\tau} = \tilde{\lambda}. \tag{25}$$

The trajectories are given by $\tilde{r} \propto \tilde{\lambda}^{\varphi}$. The crossover exponent $\varphi > 1$ can be taken to be 2 (the mean-field value for $D = 3$) without loss of generality. Under the projective change of coordinates

$$\tilde{r} = \frac{r}{1 - r - \lambda} \tag{26}$$

$$\tilde{\lambda} = \frac{\lambda}{1 - r - \lambda} \tag{27}$$

the RG equations become

$$\frac{dr}{d\tau} = r(2(1-r) - \lambda) \tag{28}$$

$$\frac{d\lambda}{d\tau} = \lambda(1 - \lambda - 2r). \tag{29}$$

Similar equations were derived in [21] from the Wegner–Houghton RG.

The advantage of coordinates (26), (27) and the preceding RG equations is that the fixed points are at finite positions, namely, the tricritical point is at $r = \lambda = 0$, the critical point is at $r = 0, \lambda = 1$ and the high-temperature Gaussian point is at $r = 1, \lambda = 0$. However, the scaling coordinates are simpler for deriving the RG potential. To do this, we must consider the RP^2 (or S^2) metric, namely,

$$ds^2 = \frac{1}{(1+\tilde{r}^2+\tilde{\lambda}^2)^2}((1+\tilde{\lambda}^2)\, d\tilde{r}^2 + (1+\tilde{r}^2)\, d\tilde{\lambda}^2 - 2\tilde{r}\tilde{\lambda}\, d\tilde{r}\, d\tilde{\lambda}).$$

We obtain

$$V = \frac{\tilde{r}^2 + \tilde{\lambda}^2/2}{1+\tilde{r}^2+\tilde{\lambda}^2} = \frac{r^2 + \lambda^2/2}{1 + 2r^2 + 2\lambda^2 - 2\lambda(1-r) - 2r}. \tag{30}$$

Since we have a minimum, a saddle point and a maximum, the Morse polynomial is $M(V) = 1 + t + t^2$. The Poincaré polynomial of RP^2 is also $P = 1 + t + t^2$ so V is a perfect Morse function.

We remark that this flow on RP^2 has three invariant subspaces RP^1, corresponding to the tricritical–critical crossover, the tricritical–Gaussian crossover and the critical–Gaussian crossover. The tricritical–critical crossover occurs for $\tilde{r} = r = 0$. The corresponding RG equations (25), (29) and potential (30) coincide with those of the $\lambda\phi^4$ theory.

It is pertinent here to relate the preceding remark with the previous remark about additional nontrivial fixed points in the $\lambda\phi^4$ theory. We see that to have more meaningful nontrivial fixed points in this theory we must introduce an additional coupling (making the coupling space two-dimensional). This is not the case for two-dimensional flows, that is, field theories with two couplings can have a fairly complicated fixed point distribution, corresponding to more complicated topologies. For example, the next more complex case than RP^2 is the torus or the Klein bottle, both corresponding to a potential with two nodes and two saddle points.

As an example of a theory with four critical points, consider a two-component theory, namely, $\lambda_1\phi_1^4 + \lambda_2\phi_2^4$, in which the two fields do not necessarily have the same dimension. The RG equations for scaling couplings are

$$\frac{d\tilde{\lambda}_1}{d\tau} = \varphi\tilde{\lambda}_1 \tag{31}$$

$$\frac{d\tilde{\lambda}_2}{d\tau} = \tilde{\lambda}_2 \tag{32}$$

equivalent to equations (24) and (25). However, it may happen that the RG equations for the renormalized couplings do not admit crossed terms; that is, in the equations corresponding to equations (28) and (29), the crossed terms are missing. This indicates that the relation between scaling and renormalized couplings consists of *independent* projective transformations for λ_1 and λ_2. Then there are three nontrivial fixed points, namely, $(\lambda_1 = 0, \lambda_2 = 1)$, $(\lambda_1 = 1, \lambda_2 = 0)$ and $(\lambda_1 = 1, \lambda_2 = 1)$; the first and the second are saddle points while the third is a node. The corresponding compactified coupling space is the direct product $RP^1 \times RP^1$, namely, the torus. The corresponding Poincaré polynomial is $P = (1+t)^2 = 1 + 2t + t^2$ and the Morse polynomial is also $1 + 2t + t^2$, so

$$V = \frac{1}{2}\left(\frac{\varphi\lambda_1^2}{(1-\lambda_1)^2+\lambda_1^2} + \frac{\lambda_2^2}{(1-\lambda_2)^2+\lambda_2^2}\right)$$

is a perfect Morse function.

7. Discussion

We have seen that the exact formulation of the RG provides us with an instrument to analyse the evolution of the infinite number of couplings of a field theory. However, this infinite-dimensional coupling space is too complex to study, except may be in the case of two-dimensional field theories [11], and implementations of the exact RG must truncate it to a finite-dimensional space of couplings [13]. The infinite number of neglected irrelevant couplings produce some uncertainty in the values of the preserved couplings, so it is necessary to add *noise* to the RG equations and, therefore, to substitute a definite location on coupling space by a probability distribution (in the context of string theory [8], this is equivalent to second quantization).

The preceding substitution of a definite location in coupling space by a probability distribution has consequences on the issue of RG irreversibility. As in the classical statistical theory of time irreversibility associated with the neglect of microscopic degrees of freedom in a macroscopic description, we have that the probability distribution in coupling space evolves irreversibly, as the corresponding Langevin or Fokker–Planck equations attest, and that the RG potential plays the role of an irreversible function, which in general has entropic nature [5, 6].

The introduction of a stochastic formulation for the RG may bring some complications but it also allows us to connect with the supersymmetric formulation of stochastic quantization. In particular, if the RG β-function is the gradient of a potential, the stochastic RG is equivalent to SUSY QM in the finite space of couplings and, hence, one can study the topology of this space by means of Morse theory with the potential and, vice versa, one can deduce the types of fixed points from the topology.

The simplest candidate for the compactification of the n-dimensional coupling space is RP^n, whose topology is well known. Hence, it is possible to deduce general patterns for RG flows. The study of the one- and two-dimensional cases shows how simple field theory RG's adapt to RP^n for $n = 1, 2$. Presumably, a generic RG flow will have the topology of the gradient flow given by a perfect Morse function on RP^n. More complex RG flows may correspond to subspaces of it, like the two-dimensional torus already described. This case is particularly interesting, because the RG equations for scaling couplings are indistinguishable from the respective equations leading to RP^2. This shows the importance of the topology, that is, how different compactifications lead to *globally* different flows. The relation between perturbative renormalization and the global character of RG flows is a subject that deserves further study.

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